

Introduction to Quantum Error Correction

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Why Quantum Error-Correction ?



- State Preparation errors
- Gate errors
- Errors due to environment (Open Systems)

The Bit-flip code



Error : $|0\rangle$ becomes $|1\rangle$ with some probability due to the effect of environment

The Bit-Flip Code



- Lets find a way to correct it

$$|0\rangle - |000\rangle$$

$$|1\rangle - |111\rangle$$

- A general state in this encoding would be

$$a|000\rangle + b|111\rangle$$

The Bit-Flip code



Possible single qubit bit-flip errors

$$a|000\rangle + b|111\rangle \longrightarrow a|001\rangle + b|110\rangle$$

$$a|000\rangle + b|111\rangle \longrightarrow a|100\rangle + b|011\rangle$$

$$a|000\rangle + b|111\rangle \longrightarrow a|010\rangle + b|101\rangle$$

The Bit-Flip code



To correct the errors we need to identify where the error has occurred
-Syndrome diagnosis

We cannot measure in the computational basis since that will destroy our state.

The Bit-Flip code



Think about these operators,

$$S_1 = Z_1 Z_2, \quad S_2 = Z_2 Z_3.$$

The general state $a|000\rangle + b|111\rangle$ is an **Eigenstate with +1 eigenvalue** of the above operators

The Bit-Flip code

But the error affected states are not $+1$ eigenstates of all the stabilizers.





- **Error on qubit 1:** $|\psi^{(1)}\rangle = X_1(\alpha|000\rangle + \beta|111\rangle) = \alpha|100\rangle + \beta|011\rangle$.

$$S_1 |\psi^{(1)}\rangle = Z_1 Z_2 (\alpha|100\rangle + \beta|011\rangle) = (-\alpha)|100\rangle + (-\beta)|011\rangle = -|\psi^{(1)}\rangle,$$

$$S_2 |\psi^{(1)}\rangle = Z_2 Z_3 (\alpha|100\rangle + \beta|011\rangle) = (+\alpha)|100\rangle + (+\beta)|011\rangle = +|\psi^{(1)}\rangle.$$

- **Error on qubit 2:** $|\psi^{(2)}\rangle = \alpha|010\rangle + \beta|101\rangle$.

$$S_1 |\psi^{(2)}\rangle = Z_1 Z_2 (\alpha|010\rangle + \beta|101\rangle) = (-\alpha)|010\rangle + (-\beta)|101\rangle = -|\psi^{(2)}\rangle,$$

$$S_2 |\psi^{(2)}\rangle = Z_2 Z_3 (\alpha|010\rangle + \beta|101\rangle) = (-\alpha)|010\rangle + (-\beta)|101\rangle = -|\psi^{(2)}\rangle.$$

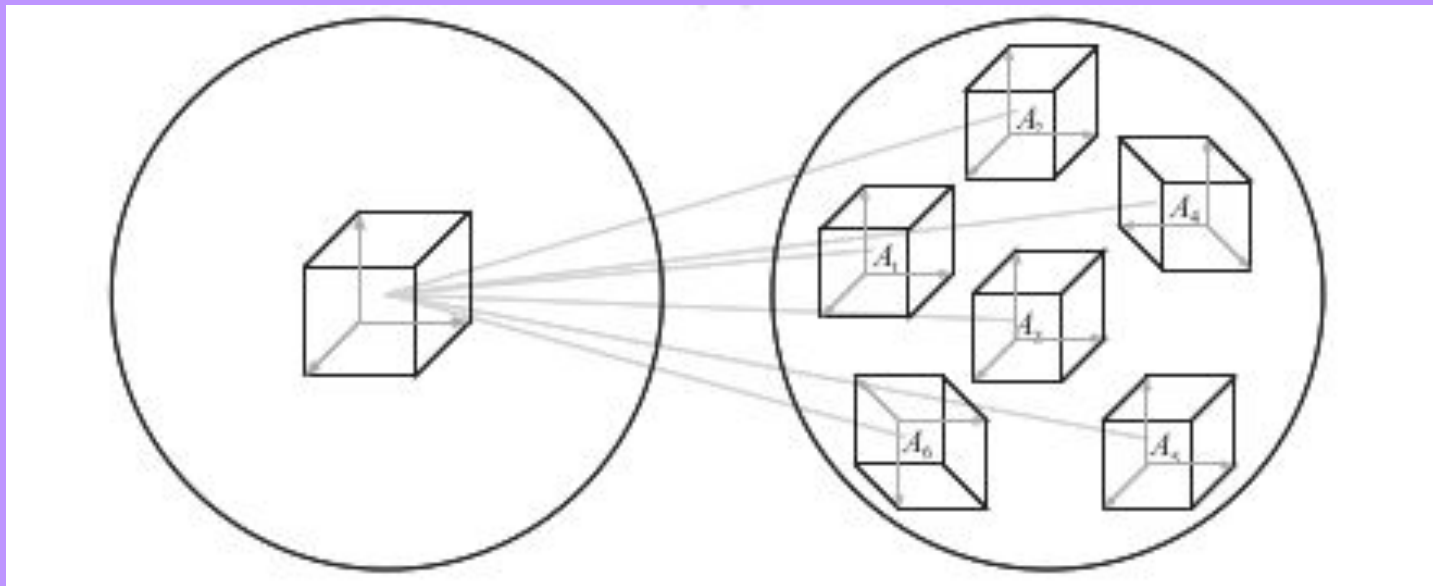


- **Error on qubit 3:** $|\psi^{(3)}\rangle = \alpha |001\rangle + \beta |110\rangle$.

$$S_1 |\psi^{(3)}\rangle = Z_1 Z_2 (\alpha |001\rangle + \beta |110\rangle) = (+\alpha) |001\rangle + (+\beta) |110\rangle = +|\psi^{(3)}\rangle,$$

$$S_2 |\psi^{(3)}\rangle = Z_2 Z_3 (\alpha |001\rangle + \beta |110\rangle) = (-\alpha) |001\rangle + (-\beta) |110\rangle = -|\psi^{(3)}\rangle.$$

The errors are taking our codespace vectors from +1 eigenspace to -1 eigenspace.



The Bit-Flip code



- Based on which stabilizer gave a -1 result we can identify on which location the error has occurred i.e on which qubit location the error has occurred
- The error correction can be done by applying an X on the location

The Bit-Flip code



- Right now we have found a way to correct bit-flip errors on a single qubit out of our three qubit encoding.
- Lets try to do something more powerful next time

Shor's 9 Qubit Code



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Shor's 9 Qubit Code



- Our encoding in this qubit is given by,

$$\begin{aligned} |0_L\rangle &= \frac{1}{2\sqrt{2}} (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle), \\ |1_L\rangle &= \frac{1}{2\sqrt{2}} (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle). \end{aligned}$$

Shor's 9 Qubit Code



- Our errors in this code are

1. Bit-Flip Error on any qubit
2. Phase-Flip Error
3. Both Bit-flip and Phase flip error

Shor's 9 Qubit Code



- The useful operators/stabilizers in this code are

Name	Operator
g_1	$ZZIIIIIII$
g_2	$IZZIIIIII$
g_3	$III ZZIIII$
g_4	$IIII ZZIII$
g_5	$IIII IIZZ I$
g_6	$IIII I I I ZZ$
g_7	$XXXXXX III$
g_8	$III XXXXXX$
$\tilde{Z}$	$XXXXXXXXXX$
$\tilde{X}$	$ZZZZZZZZZZ$

Shor's 9 Qubit Code



- The code-space is as expected a positive eigenspace of all stabilizers

Name	Operator
g_1	$ZZIIIIIII$
g_2	$IZZIIIIII$
g_3	$III ZZIIII$
g_4	$IIII ZZIII$
g_5	$IIII II ZZI$
g_6	$IIII III ZZ$
g_7	$XXXXX III$
g_8	$III XXXXXX$
$\tilde{Z}$	$XXXXXXXXX$
$\tilde{X}$	$ZZZZZZZZZ$

Shor's 9 Qubit Code

- The errors again as expected will take the codespace to the -1 eigenspace of some stabilizers



Shor's 9 Qubit Code



Error X_2 (in first block):

Stabilizer	Effect	Eigenvalue
$Z_1 Z_2$	touches qubit 2 → anticommutes	-1
$Z_2 Z_3$	touches qubit 2 → anticommutes	-1
others ($Z_4 Z_5, \dots$)	commute	+1
$X_1 \dots X_6, X_4 \dots X_9$	commute	+1

So the two Z -type stabilizers in that block give -1 , uniquely identifying the flipped qubit.

Shor's 9 Qubit Code



Error: X_7

Stabilizer

Eigenvalue

$$Z_1 Z_2$$

+1

$$Z_2 Z_3$$

+1

$$Z_4 Z_5$$

+1

$$Z_5 Z_6$$

+1

$$Z_7 Z_8$$

-1

$$Z_8 Z_9$$

+1

$$X_1 X_2 X_3 X_4 X_5 X_6$$

+1

$$X_4 X_5 X_6 X_7 X_8 X_9$$

+1

Shor's 9 Qubit Code



Block with Z error	$X_1 \dots X_6$	$X_4 \dots X_9$
(1,2,3)	-1	+1
(4,5,6)	-1	-1
(7,8,9)	+1	-1

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- Once detected we can again correct X errors by applying X_i on the correct location and Z errors by applying $Z_i Z_{i+1} Z_{i+2}$.
- Both Bit-flip and phase flip errors (XZ) error can be corrected doing it step-wise

Shor's 9 Qubit Code



- Any Unitary U can be written as,

$$E = I/2 + aX + bY + cZ$$

- $E1 |\psi\rangle$ can thus be written as a superposition of four terms, $|\psi\rangle$, $X1 |\psi\rangle$, $Z1 |\psi\rangle$, $X1 Z1 |\psi\rangle$.

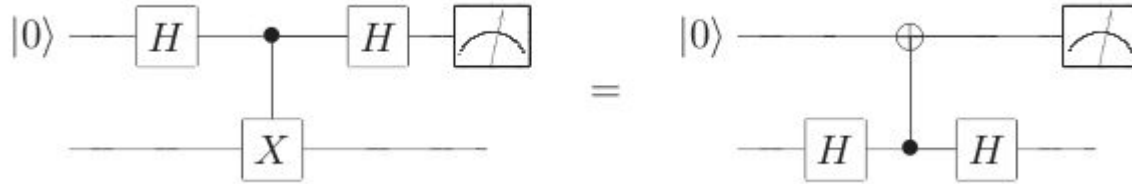
Shor's 9 Qubit Code



Every error U is broken down to either X error or Z error or XZ errors

- Once we correct these three we can **correct any arbitrary unitary error.**

How to measure a state in that operator basis



1) Initial state

$$|0\rangle_a \otimes (\alpha|0\rangle + \beta|1\rangle)_b.$$

2) Apply X on bottom

$$X|\psi\rangle_b = \alpha|1\rangle + \beta|0\rangle \quad \Rightarrow \quad |0\rangle_a \otimes (\alpha|1\rangle + \beta|0\rangle)_b.$$

3) Apply H on top

$$\begin{aligned} H|0\rangle_a &= |+\rangle_a = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \\ \Rightarrow \quad |+\rangle_a &\otimes (\alpha|1\rangle + \beta|0\rangle)_b. \end{aligned}$$

4) Apply CNOT (control=top, target=bottom)

$$\text{CNOT} : |0\rangle \otimes |\phi\rangle \mapsto |0\rangle \otimes |\phi\rangle, \quad |1\rangle \otimes |\phi\rangle \mapsto |1\rangle \otimes X|\phi\rangle.$$

Hence

$$\begin{aligned} &\frac{1}{\sqrt{2}} \left(|0\rangle_a \otimes (\alpha|1\rangle + \beta|0\rangle)_b + |1\rangle_a \otimes X(\alpha|1\rangle + \beta|0\rangle)_b \right) \\ &= \frac{1}{\sqrt{2}} \left(|0\rangle_a \otimes (\alpha|1\rangle + \beta|0\rangle)_b + |1\rangle_a \otimes (\alpha|0\rangle + \beta|1\rangle)_b \right). \end{aligned}$$

5) Apply the final H on top

Use $H|0\rangle = |+\rangle$, $H|1\rangle = |-\rangle$:

$$|\Psi_{\text{final}}\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_a \otimes (\alpha|1\rangle + \beta|0\rangle)_b + |-\rangle_a \otimes (\alpha|0\rangle + \beta|1\rangle)_b \right).$$

Equivalently, in the computational basis of the ancilla,

$$\begin{aligned} |\Psi_{\text{final}}\rangle &= \frac{1}{2} \left[|0\rangle_a \otimes ((\alpha + \beta)|0\rangle + (\alpha + \beta)|1\rangle)_b \right. \\ &\quad \left. + |1\rangle_a \otimes ((\beta - \alpha)|0\rangle + (\alpha - \beta)|1\rangle)_b \right]. \end{aligned}$$